Exercise 61

(a) Sketch the graph of the function $f(x) = x|x|$.

(b) For what values of x is f differentiable?

(c) Find a formula for f'.

Solution

Use the definition of the derivative to find f'.

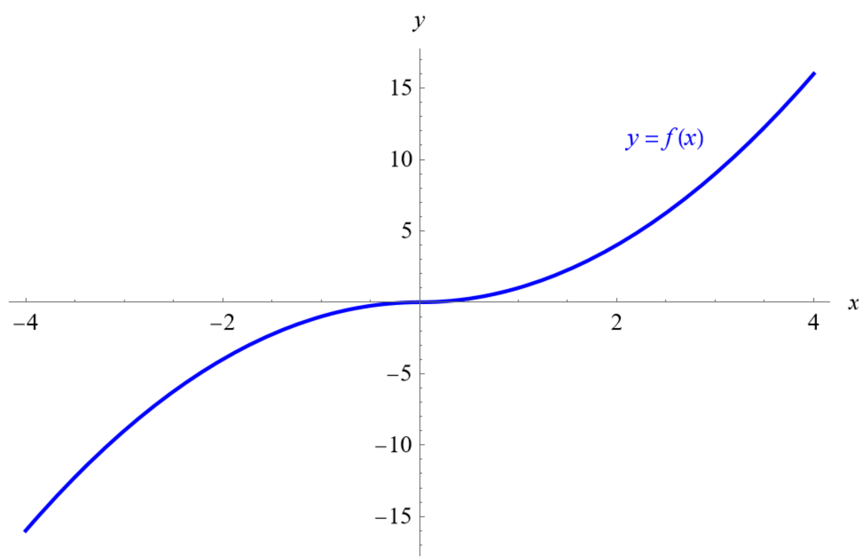
$$
\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)|x+h| - x|x|}{h} \\
&= \lim_{h\to 0} \frac{(x+h)\sqrt{(x+h)^2} - x\sqrt{x^2}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)\sqrt{(x+h)^2} - x\sqrt{x^2}}{h} \cdot \frac{(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}}{(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}} \\
&= \lim_{h\to 0} \frac{(x+h)^2(x+h)^2 - x^2(x^2)}{h\left[(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}\right]} \\
&= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h\left[(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}\right]} \\
&= \lim_{h\to 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h\left[(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}\right]} \\
&= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h\left[(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}\right]} \\
&= \lim_{h\to 0} \frac{4x^3 + 6x^2h + 4xh^2 + h^3}{(x+h)\sqrt{(x+h)^2} + x\sqrt{x^2}} \\
&= \frac{4x^3}{(x)\sqrt{(x)^2} + x\sqrt{x^2}} \\
&= \frac{4x^3}{2x\sqrt{x^2}} \\
&= \frac{2x^2}{\sqrt{x^2}} \\
&= \frac{2x^2}{|x|} = \frac{2|x|^2}{|x|} \\
&= 2x \operatorname{sgn} x = 2|x|
\end{aligned}
$$

$f(x) = x|x|$ is not differentiable at 0 because there's $|x|$ in the denominator, and for any rational function the denominator cannot be zero.

$$|x| \neq 0$$

$$x \neq 0$$

The domain of $f'(x)$ is $\{x \mid x \neq 0\}$. Below is a graph of $f(x)$ versus x.



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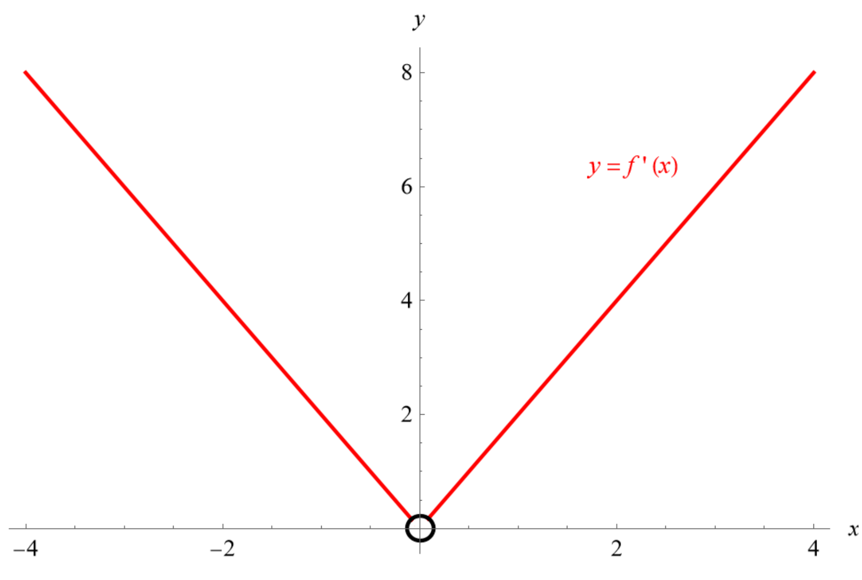

This answer is in disagreement with the one at the back of the book.